

OSEEN SYSTEM WITH CORIOLIS TERM

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ABSTRACT

This paper is devoted to solutions of the Dirichlet problem for the Oseen system with Coriolis term $-\Delta u(z) + \tau \partial_1 u(z) - (\omega \times z) \cdot \nabla u(z) + \omega \times u(z) + \nabla p(z) = f(z)$, $\nabla \cdot u$ in Ω , $u = g$ on $\partial\Omega$ in the homogeneous Sobolev space $\tilde{W}^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega)$ with $2 \leq q < 3$. Here $\Omega \subset \mathbb{R}^3$ is an exterior domain. Kracmar, Necasová and Penel proved that if Ω has boundary of class C^2 , $g \equiv 0$ and $f \in D^{-1,q}(\Omega; \mathbb{R}^3)$, then there exists a unique solution of the problem. This paper shows that this result holds true even for domains with Lipschitz boundary. Moreover, we prove unique solvability of the problem for general $g \in W^{1-1/q,q}(\partial\Omega; \mathbb{R}^3)$ and $f \in D^{-1,q}(\Omega; \mathbb{R}^3)$.

Key words: Oseen system with Coriolis term.

1. INTRODUCTION

This paper is devoted to the Dirichlet problem for the Oseen system with Coriolis term

$$-\Delta u(z) + r \partial_1 u(z) - (\omega \times z) \cdot \nabla u(z) + \omega \times u(z) + \nabla p(z) = f(z), \text{ in } \Omega \quad (1)$$

$$\nabla \cdot u = 0 \text{ in } \Omega, \quad (2)$$

$$u = g \text{ on } \partial\Omega \quad (3)$$

Here $\Omega \subset \mathbb{R}^3$ is an unbounded domain with compact Lipschitz boundary, $\omega = (\rho, 0, 0)$ and $\tau, \rho \in \mathbb{R}^1 \setminus \{0\}$. This problem arises by linearization and normalization of a mathematical model describing the stationary flow of a viscous incompressible fluid around a rigid body moving at a constant velocity and rotating at a constant angular velocity, under the assumption that the velocity of the body and its angular velocity are parallel to each other. More information on physical background can be found in G.P. Galdi(2002) or Š. Necasová *et al.* (2016).

The problem (1)–(3) has been studied by many authors. (See for example R. Farwig *et al.*(2007), R. Farwig *et al.*(2011), R. Farwig *et al.*(2008), R. Farwig

and J. Neustupa (2007), H. Heck *et al.* (2012), S. Kracmar *et al.* (JMSJ 2010), M. Kyed (QAM 2013), M. Kyed (DCDS 2013).) The papers R. Farwig *et al.* (2009), G.P. Galdi (2011), G.P. Galdi *et al.* (2007), D. Kim (2018) and M. Kyed (2014) studied the problem in homogeneous Sobolev spaces $D^{2,q}(\Omega; \mathbb{R}^3) \times D^{1,q}(\Omega)$. The papers R. Farwig *et al.* (2011), G.P. Galdi (2011), T. Hishida (2006), D. Kim (2018), D. Kim (2019) and S. Kracmar *et al.* (QAM 2010) are devoted to solutions of the problem in $D^{1,q}(\Omega; \mathbb{R}^3) \times L^q_{loc}(\Omega)$. Very weak solutions of the problem with $u \in L^q(\Omega; \mathbb{R}^3)$ are in D. Kim (2018).

We are going to study the problem in $D^{1,q}(\Omega; \mathbb{R}^3) \times L^q_{loc}(\Omega)$. Let us gather the known results. D. Kim writes in D. Kim (2018): Let W be an unbounded domain with compact smooth boundary. If $g \in H^{1/2}(W; \mathbb{R}^3)$ and $f \equiv 0$, then there exists a unique solution $(u, p) \in [D^{1,2}(W; \mathbb{R}^3) \cap L^4(W; \mathbb{R}^3)] \times L^2(W)$ of (1)–(3). G.P. Galdi proves in G.P. Galdi (2011): Let W be an unbounded domain with compact Lipschitz boundary. Let $g \in H^{1/2}(\partial W, \mathbb{R}^3)$, $f \in D^{-1,2}(W, \mathbb{R}^3)$. Then there exists at least one solution $(u, p) \in D^{1,2}(W, \mathbb{R}^3) \times L^2_{loc}(W)$ of (1)–(3) such that

$$\int_{\partial B(0;1)} |u(rx)| d\sigma(x) = o(r^{-1/2}) \text{ as } r \rightarrow \infty.$$

D Kim shows in D. Kim (2019): Let Ω have smooth boundary. Suppose that $f = f_1 + f_2$, $f_1 \in D^{-1,2}(\Omega; \mathbb{R}^3)$, $f_2 \in L^{4/3}(\Omega; \mathbb{R}^3)$ and $g \in W^{1/2,2}(\Omega; \mathbb{R}^3)$. Then there exists a unique solution $u \in D^{1,2}(\Omega; \mathbb{R}^3) \cap L^4(\Omega; \mathbb{R}^3)$ of (1)–(3) such that

$$u(x) \rightarrow 0, |x| \rightarrow \infty.$$

S. Kracmar, Š. Necasová and P. Penel proved in S. Kracmar *et al.* (QAM 2010) the following result: Let Ω be an unbounded domain with compact boundary of class C^2 , $g \equiv 0$ and $f \in D^{-1,q}(\Omega, \mathbb{R}^3)$ with $3/2 < q < 3$. Then there exists a unique solution $(u, p) \in D^{1,q}_0(\Omega, \mathbb{R}^3) \times L^q(\Omega)$ of (1), (2). (That means that (u, p) is a solution of the problem (1)–(3) with $g \equiv 0$ in $\tilde{W}^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega)$.)

This paper studies the problem (1)–(3) for exterior domains with Lipschitz boundary. It is shown that there exists a unique solution (u, p) in the space $\tilde{W}^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega)$ of (1)–(3) for $f \in D^{-1,q}(\Omega; \mathbb{R}^3)$ and $g \in W^{1-1/q,q}(\partial\Omega; \mathbb{R}^3)$ with $2 \leq q < 3$. Remark that $u \in D^{1,q}_0(\Omega, \mathbb{R}^3)$ if and only if $u \in \tilde{W}^{1,q}(\Omega; \mathbb{R}^3)$ and the trace of u is zero, *i.e.* if $g \equiv 0$. So, our result is a generalization of the result of S. Kracmar *et al.* (QAM 2010) under the assumption that $2 \leq q < 3$. (For the definition of function spaces see the next section.)

2. FUNCTION SPACES

Let $\Omega \subset \mathbb{R}^m$ be a domain (*i.e.* an open connected set), $1 \leq q \leq \infty$ and $k \in \mathbb{N}$. We denote by $W^{k,q}(\Omega)$ the classical Sobolev space, *i.e.* $W^{k,q}(\Omega) := \{u \in L^q(\Omega); \partial^\alpha u$

$\in L^q(\Omega) \forall |\alpha| \leq k$. If $M = \Omega$ or $M = \bar{\Omega}$ we denote by $W_{loc}^{k,q}(M)$ the set of all functions u on Ω such that $u \in W^{k,q}(G)$ for each bounded open set G with $\bar{G} \subset M$. Denote by $C_c^\infty(\Omega)$ the space of infinitely differentiable functions with compact support in Ω and by $W_0^{k,q}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W^{k,q}(\Omega)$. Further denote by $W^{-k,q}(\Omega)$ the dual space of $W_0^{k,q'}(\Omega)$ with $q' = q/(q-1)$.

The homogeneous Sobolev space $D^{k,q}(\Omega)$ consists of distributions on Ω with derivatives of the order k in $L^q(\Omega)$. If $u \in D^{k,q}(\Omega)$ then $u \in W_{loc}^{k,q}(\Omega)$ by §1.2.3 in V.G. Maz'ya *et al.* (1997). If Ω has compact Lipschitz boundary then $D^{k,q}(\Omega) \subset W_{loc}^{k,q}(\bar{\Omega})$ (see Proposition 1.25.2 in D. Medková (2018)).

Fix a bounded open set G such that $\bar{G} \subset \Omega$. Then $D^{k,q}(\Omega)$ is a Banach space with the norm

$$\|u\|_{D^{k,q}(\Omega)} := \|u\|_{L^q(G)} + \|\nabla^k u\|_{L^q(\Omega)}.$$

Moreover, different choices of G give equivalent norms. (See §1.5.3, Corollary 2 in V.G. Maz'ya *et al.* (1997).) If Ω is a bounded domain with Lipschitz boundary then $D^{k,q}(\Omega) = W^{k,q}(\Omega)$ and the corresponding norms are equivalent. (See §1.5.2-§1.5.4 in V.G. Maz'ya *et al.* (1997)).

Denote by $D_0^{k,q}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $D^{k,q}(\Omega)$. If $\bar{\Omega} \neq \mathbb{R}^m$ then $\|\nabla \varphi\|_{L^q(\Omega)}$ is an equivalent norm in $D^{1,q}(\Omega)$. (See Lemma 1.25.4 in D. Medková (2018).) Denote by $D^{-1,q}(\Omega)$ the dual space of $D^{1,q'}(\Omega)$ with $q' = q/(q-1)$.

Denote $\tilde{W}^{k,q}(\Omega) := \{u|_{\Omega}; u \in D_0^{k,q}(\mathbb{R}^m)\}$. If Ω has compact Lipschitz boundary then $\tilde{W}^{k,q}(\Omega)$ is the closure of $C_c^\infty(\mathbb{R}^m)$ in $D^{k,q}(\Omega)$.

Suppose now that Ω has compact Lipschitz boundary. If $0 < s < 1$ and $1 < q < \infty$ we denote by $W^{s,q}(\partial\Omega)$ the set of all $f \in L^q(\partial\Omega)$ with $\|f\|_{W^{s,q}(\partial\Omega)} < \infty$, where

$$\|f\|_{W^{s,q}(\partial\Omega)} := \sqrt[q]{\|f\|_{L^q(\partial\Omega)}^q + \int_{\partial\Omega \times \partial\Omega} \frac{|f(x) - f(y)|^q}{|x - y|^{m-1+qs}} d\sigma^*(x) d\sigma(y)}$$

Then there exists a unique linear continuous mapping γ_Ω (called trace mapping) from $W^{1,q}(\Omega)$ onto $W^{1-1/q,q}(\partial\Omega)$ such that $\gamma_\Omega u = u$ on $\partial\Omega$ for all $u \in W^{1,q}(\Omega) \cap C^{0,1}(\bar{\Omega})$, where $C^{0,1}(\bar{\Omega})$ is the space of Lipschitz continuous functions on $\bar{\Omega}$. (See Theorem 1.5.1.2 in P. Grisvard (2011).) Similarly, there is a unique linear continuous mapping $\gamma_\Omega : D^{1,q}(\Omega) \rightarrow W^{1-1/q,q}(\partial\Omega)$ such that $\gamma_\Omega u = u$ on $\partial\Omega$ for all $u \in D^{1,q}(\Omega) \cap C^{0,1}(\bar{\Omega})$.

The following lemmas are well known. We prove them for the lack of references.

Lemma 2.1: Let $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary. Suppose that $1 < r < q < \infty$. Then $W^{-1/q,q}(\partial\Omega) \rightarrow W^{-1/r,r}(\partial\Omega)$.

Proof: Without loss of generality we can suppose that Ω is bounded. According to Theorem 1.5.1.2 in *P. Grisvard (2011)* there exists a bounded operator $E : W^{-1/q,q}(\partial\Omega) \rightarrow W^{1,q}(\Omega)$ such that $\gamma_\Omega Eu = u$ for all $u \in W^{-1/q,q}(\partial\Omega)$. Hölder's inequality gives $W^{1,q}(\Omega) \rightarrow W^{1,r}(\Omega)$. Since the trace $\gamma_\Omega : W^{1,r}(\Omega) \rightarrow W^{-1/r,r}(\partial\Omega)$ is continuous and $\gamma_\Omega Eu = u$ for $u \in W^{-1/q,q}(\partial\Omega)$, we infer $W^{-1/q,q}(\partial\Omega) \rightarrow W^{-1/r,r}(\partial\Omega)$.

Lemma 2.2: Let $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary. Suppose that $1 < r < q < \infty$. Then $W^{-1/q,q}(\partial\Omega) \rightarrow W^{-1/r,r}(\partial\Omega)$.

Proof: Denote $q' = q/(q-1)$ and $r' = r/(r-1)$. Since $q' < r'$ Lemma 2.1 forces $W^{-1/r',r'}(\partial\Omega) \rightarrow W^{-1/q',q'}(\partial\Omega)$. As $1/q = 1 - 1/q'$ and $1/r = 1 - 1/r'$ we infer

$$\begin{aligned} W^{-1/q,q}(\partial\Omega) &= [W^{1/q,q'}(\partial\Omega)]' = [W^{1-1/q',q'}(\partial\Omega)]' \rightarrow [W^{1-1/r',r'}(\partial\Omega)]' \\ &= [W^{1/r,r'}(\partial\Omega)]' = W^{-1/r,r}(\partial\Omega). \end{aligned}$$

Proposition 2.3: Let $\Omega \subset \mathbb{R}^m$ be an unbounded domain with compact Lipschitz boundary or $\Omega = \mathbb{R}^m$. Suppose that $1 < q < m$. Denote by $P_0(\mathbb{R}^m)$ the space of constant functions in \mathbb{R}^m . Then $D^{1,q}(\Omega) = \tilde{W}^{1,q}(\Omega) \oplus P_0(\Omega)$ and $\tilde{W}^{1,q}(\Omega)$ is formed by $u \in D^{1,q}(\Omega)$ such that

$$\lim_{r \rightarrow \infty} \int_{\partial B(0;1)} |u(rx)| d\sigma(x) = 0$$

where $B(z; r) = \{x \in \mathbb{R}^m; |z - x| < r\}$. If $u \in \tilde{W}^{1,q}(\Omega)$ then $u \in L^{m/(m-q)}(\Omega)$.

(See Lemma 18 in *D. Medková (2019)*, Proposition 3.38.5 and Proposition 3.38.4 in *D. Medková (2018)*.)

Remark that the claims of Proposition 2.3 stop to hold for $q \geq m$.

3. DIRICHLET PROBLEM IN A BOUNDED DOMAIN

In this section we study the Dirichlet problem in a bounded domain. We begin with the Dirichlet problem for the homogeneous Stokes system.

Proposition 3.1: Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary. Denote by n^Ω the unit exterior normal of Ω . Let $2 \leq q < 3$ and $g \in W^{-1/q,q}(\partial\Omega; \mathbb{R}^3)$. Then there exists a solution $(u, p) \in W^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega)$ of

$$-\Delta u + \nabla p = 0, \quad \nabla \cdot u = 0 \text{ in } \Omega \quad (4)$$

with the Dirichlet condition (3) if and only if

$$\int_{\partial\Omega} n^\Omega \cdot g d\sigma = 0. \quad (5)$$

A velocity u is unique, a pressure p is unique up to an additive constant.

Proof: If $(u, p) \in W^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega)$ is a solution of (4), (3), then Green's formula gives

$$\int_{\partial\Omega} n^\Omega \cdot g d\sigma = \int_{\partial\Omega} \nabla \cdot u dx = 0.$$

Let $(u, p) \in W^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega)$ be a solution of (4), (3) with $g \equiv 0$. Hölder's inequality gives that $(u, p) \in W^{1,2}(\Omega; \mathbb{R}^3) \times L^2(\Omega)$. So, $u \equiv 0$ and p is constant by Theorem IV.1.1 in G.P. Galdi (2011).

We now show the existence of a solution. We use the integral equation method. Denote by $E = (E_{ij})$ the velocity part of the fundamental solution for the Stokes system and by $Q = (Q_j)$ the pressure part of this fundamental solution. Here

$$E_{ij}(x) = \frac{1}{8\pi} \left(\delta_{ij} \frac{1}{|x|} + \frac{x_i x_j}{|x|^3} \right), \quad Q_j(x) = \frac{x_j}{4\pi|x|^3}.$$

where $x \in \mathbb{R}^3 \setminus \{0\}$ and $i, j = 1, 2, 3$. If $f \in W^{-1/q,q}(\partial\Omega; \mathbb{R}^3)$ then we define the velocity part of the single layer potential with the density f by

$$E_\Omega f(x) := \langle f, E(x - \cdot) \rangle, \quad x \in \mathbb{R}^3 \setminus \partial\Omega$$

and the pressure part of the single layer potential with the density f by

$$Q_\Omega f(x) := \langle f, Q(x - \cdot) \rangle, \quad x \in \mathbb{R}^3 \setminus \partial\Omega.$$

If $u = E_\Omega f$ and $p = Q_\Omega f$ then (u, p) is a classical solution of (4). (See for example M. Mitrea *et al.* (2012).) Moreover, $E_\Omega f \in W^{1,q}(\Omega; \mathbb{R}^3)$. (We can consider Ω as a subset of a compact manifold and use Theorem 3.1 in M. Mitrea *et al.* (2001).) Define $\tilde{f} \in W^{-1,q}(R^3; R^3)$ by

$$\langle \tilde{f}, v \rangle := \langle f, \gamma_\Omega v \rangle, \quad v \in W^{1,q}(R^3; R^3).$$

Since $Q_\Omega f = Q * \tilde{f}$, Lemma 8 in D. Medková (2019) gives that $Q_\Omega f \in L^q(\Omega)$. Denote $\mathcal{E}_\Omega f := \gamma_\Omega E_\Omega f$. According to Theorem 10.5.3 in M. Mitrea *et al.* (2012)

$$\mathcal{E}_\Omega : B_{-1/q}^{q,q}(\partial\Omega; \mathbb{R}^3) \rightarrow B_{1-1/q}^{q,q}(\partial\Omega; \mathbb{R}^3)$$

is a Fredholm operator with index 0 between these Besov spaces. Remember that $B_{-1/q}^{q,q}(\partial\Omega; \mathbb{R}^3) = W^{-1/q,q}(\partial\Omega; R^3)$ and $B_{1-1/q}^{q,q}(\partial\Omega; \mathbb{R}^3) = W^{1-1/q,q}(\partial\Omega; \mathbb{R}^3)$. (See Proposition 2.5.1 in M. Mitrea *et al.* (2012), Proposition 2.52 in I. Mitrea *et al.* (2013) and Lemma 36.1 in L. Tartar (2007).)

Denote by C_1, \dots, C_k all bounded components of $R^3 \setminus \bar{\Omega}$. Fix $z^j \in C_j$ for $j = 1, \dots, k$. Put

$$w_j(x) := \frac{x - z^j}{|x - z^j|^3}.$$

Define $\psi_j(x) := |x - z_j|^{-1}$. Then $\Delta\psi_j = 0$ in Ω by Remark 2.1.3 in D. Medková (2018). Since $w_j = \nabla\psi_j$ we have $\Delta w_j = \nabla(\Delta\psi_j) = 0$ and $\nabla \cdot w_j = \Delta\psi_j = 0$ in Ω . For $f \in W^{-1/q,q}(\partial\Omega, \mathbb{R}^3)$ define

$$\tilde{E}_\Omega f = E_\Omega f + \sum_{j=1}^k w_j \langle f, w_j \rangle.$$

Then $\tilde{E}_\Omega f \in W^{1,q}(\Omega, \mathbb{R}^3)$. If $\mathbb{R}^3 \setminus \Omega$ is connected then $\tilde{E}_\Omega f = E_\Omega f$. Since $\Delta w_j = 0$ and $\nabla \cdot w_j = 0$ in Ω , we obtain

$$-\Delta \tilde{E}_\Omega f + \nabla Q_\Omega f = 0, \quad \nabla \cdot \tilde{E}_\Omega f = 0 \text{ in } \Omega.$$

Denote $\tilde{\mathcal{E}}_\Omega f := \gamma_\Omega \tilde{E}_\Omega f$. Then $(u, p) = (\tilde{E}_\Omega f, Q_\Omega f)$ is a solution of (4), (3) if and only if $\tilde{\mathcal{E}}_\Omega f = g$. Since $\tilde{\mathcal{E}}_\Omega - \mathcal{E}_\Omega : W^{-1/q,q}(\partial\Omega; \mathbb{R}^3) \rightarrow W^{1-1/q,q}(\partial\Omega; \mathbb{R}^3)$ is finite-dimensional and therefore compact, the operator $\tilde{\mathcal{E}}_\Omega : W^{-1/q,q}(\partial\Omega; \mathbb{R}^3) \rightarrow W^{1-1/q,q}(\partial\Omega; \mathbb{R}^3)$ is Fredholm with index 0. According to Proposition 4 in D. Medková *et al.* (2015) we have

$$\tilde{\mathcal{E}}_\Omega(W^{-1/2,2}(\partial\Omega, \mathbb{R}^3)) = \left\{ g \in W^{1/2,2}(\partial\Omega, \mathbb{R}^3); \int_{\partial\Omega} g \cdot n^\Omega = 0 \right\}.$$

Since $\tilde{\mathcal{E}}_\Omega : W^{-1/2,2}(\partial\Omega; \mathbb{R}^3) \rightarrow W^{1-1/2,2}(\partial\Omega; \mathbb{R}^3)$ is a Fredholm operator with index 0, the dimension of its kernel is equal to 1. Since $W^{-1/q,q}(\partial\Omega; \mathbb{R}^3) \subset W^{-1/2,2}(\partial\Omega; \mathbb{R}^3)$ by Lemma 2.2, the dimension of the kernel of the operator $\tilde{\mathcal{E}}_\Omega : W^{-1/q,q}(\partial\Omega; \mathbb{R}^3) \rightarrow W^{1-1/q,q}(\partial\Omega; \mathbb{R}^3)$ is at most 1. Since

$$\int_{\partial\Omega} n^\Omega \cdot \tilde{\mathcal{E}}_\Omega f d\sigma = 0$$

by (5), we infer that

$$\tilde{\mathcal{E}}_\Omega(W^{-1/q,q}(\partial\Omega, \mathbb{R}^3)) = \left\{ g \in W^{1-1/1,q,q}(\partial\Omega, \mathbb{R}^3); \int_{\partial\Omega} g \cdot n^\Omega = 0 \right\}.$$

So, if $g \in W^{1-1/q,q}(\partial\Omega, \mathbb{R}^3)$ satisfies (5), then there exists a solution $(u, p) \in W^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega)$ of (4).

Now we are able to study the Oseen system with Coriolis term.

Theorem 3.2: Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary. Suppose that $\tau, \rho \in \mathbb{R}^1$, $\omega = (\rho, 0, 0)$, $f \in W^{-1,q}(\Omega; \mathbb{R}^3)$, $g \in W^{1-1/q,q}(\partial\Omega; \mathbb{R}^3)$ and $h \in L^q(\Omega)$ with $2 \leq q < 3$. Then there exists a solution $(u, p) \in W^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega)$ of

$$-\Delta u(z) + \tau \partial_1 u(z) - (\omega \times z) \cdot \nabla u(z) + \omega \times u(z) + \nabla p(z) = f(z) \text{ in } \Omega, \quad (6)$$

$$\nabla \cdot u = h \text{ in } \Omega \quad (7)$$

with the Dirichlet condition (3) if and only if

$$\int_{\partial\Omega} n^\Omega \cdot g d\sigma = \int_{\Omega} h dx. \quad (8)$$

A velocity u is unique, a pressure p is unique up to an additive constant.

Proof: If $(u, p) \in W^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega)$ is a solution of (7), (3) then Green's formula gives

$$\int_{\partial\Omega} n^\Omega \cdot g d\sigma = \int_{\Omega} \nabla \cdot u dx = \int_{\Omega} h dx.$$

Suppose now that $(u, p) \in W^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega)$ is a solution of (6), (7), (3) with $h \equiv 0$, $f \equiv 0$ and $g \equiv 0$. Hölder's inequality forces that $(u, p) \in W^{1,2}(\Omega; \mathbb{R}^3) \times L^2(\Omega)$. Since

$$-\Delta u(z) + \partial_1 u(z) - (\omega \times z) \cdot \nabla u(z) + \omega \times u(z) + \nabla p(z) = (1 - \tau) \partial_1 u(z) \in L^2(\Omega),$$

Lemma 2.3 in R. Farwig and J. Neustupa (2007) forces that (u, p) is in the space $W_{loc}^{2,2}(\Omega; \mathbb{R}^3) \times W_{loc}^{1,2}(\Omega)$. Fix $v \in C_c^\infty(\Omega; \mathbb{R}^3)$. Choose a domain G with smooth boundary such that v is supported in G and $\bar{G} \subset \Omega$. Since $v = 0$ on ∂G , Green's formula gives

$$\begin{aligned} 0 &= \int_{\partial G} v \cdot \left\{ \frac{\partial u}{\partial n} - \frac{1}{2} r_1 u + \frac{1}{2} [(\omega \times z) \cdot n^G] u - p n^G \right\} d\sigma(z) \\ &= \int_G \{ \nabla v \cdot \nabla u + v \cdot \Delta u - \frac{1}{2} \tau v \partial_1 u - \frac{1}{2} \tau u \partial_1 v + \frac{1}{2} v \cdot [(\omega \times z) \cdot \nabla u] \\ &\quad + \frac{1}{2} u \cdot [(\omega \times z) \cdot \nabla v] - v \cdot \nabla p - p(\nabla \cdot v) \} dz \\ &= \int_G \{ -v \cdot [-\Delta u(z) + r \partial_1 u(z) - (\omega \times z) \cdot \nabla u(z) + \omega \times u(z) + \nabla p(z)] + \nabla v \cdot \nabla u \\ &\quad + \frac{1}{2} \tau v \partial_1 u - \frac{1}{2} \tau u \partial_1 v - \frac{1}{2} v \cdot [(\omega \times z) \cdot \nabla u] + \frac{1}{2} u \cdot [(\omega \times z) \cdot \nabla v] + v \cdot (\omega \times u) \\ &\quad - p(\nabla \cdot v) \} dz \\ &= \int_{\Omega} \{ \nabla v \cdot \nabla u + \frac{1}{2} \tau v \partial_1 u - \frac{1}{2} \tau u \partial_1 v - \frac{1}{2} v \cdot [(\omega \times z) \cdot \nabla u] + \frac{1}{2} u \cdot [(\omega \times z) \cdot \nabla v] \\ &\quad + v \cdot (\omega \times u) - p(\nabla \cdot v) \} dz. \end{aligned}$$

Since $u = 0$ on $\partial\Omega$ there exists a sequence $\{v_n\} \subset C_c^\infty(\Omega; \mathbb{R}^3)$ such that $v_n \rightarrow u$ in $W^{1,2}(\Omega; \mathbb{R}^3)$. (See Theorem 6.6.4 in A. Kufner *et al.* (1977).) So,

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \int_{\Omega} \left\{ \nabla v_n \cdot \nabla u + \frac{1}{2} \tau v_n \partial_1 u - \frac{1}{2} \tau u \partial_1 v_n - \frac{1}{2} v_n \cdot [(\omega \times z) \cdot \nabla u] + \frac{1}{2} u \cdot [(\omega \times z) \cdot \nabla v_n] \right. \\ &\quad \left. + v_n \cdot (\omega \times u) - p(\nabla \cdot v_n) \right\} dz \\ &= \int_{\Omega} \left\{ \nabla v_n \cdot \nabla u + \frac{1}{2} \tau u \partial_1 u - \frac{1}{2} \tau u \partial_1 v_n - \frac{1}{2} u \cdot [(\omega \times z) \cdot \nabla u] + \frac{1}{2} u \cdot [(\omega \times z) \cdot \nabla v_n] \right. \\ &\quad \left. + u \cdot (\omega \times u) - p(\nabla \cdot u) \right\} dz \\ &= \int_{\Omega} \left[|\nabla u|^2 + u \cdot (\omega \times u) \right] dz. \end{aligned}$$

Since $\omega \times u(z)$ is orthogonal to $u(z)$, we infer that

$$0 = \int_{\Omega} |\nabla u|^2 dz.$$

Since $\nabla u \equiv 0$, the velocity u is constant. As $u = 0$ on $\partial\Omega$, we deduce that $u \equiv 0$. Therefore $\nabla p(z) = \Delta u(z) - \tau \partial_1 u(z) + (\omega \times z) \cdot \nabla u(z) - \omega \times u(z) = 0$ and the pressure p is constant.

Now we show the existence of a solution of the problem under the condition (8). We begin with the Stokes system, *i.e.* with $\tau = \rho = 0$. Suppose that the condition (8) is satisfied. Choose a bounded domain $D \subset \mathbb{R}^3$ with smooth boundary such that $\bar{\Omega} \subset D$. We can consider $W_0^{1,q'}(\Omega)$ to be a closed subspace of $W_0^{1,q'}(D)$, where $q' = q/(q-1)$. According to the Hahn-Banach theorem there exists $F \in W^{-1,q}(D; \mathbb{R}^3)$ such that $\langle F, v \rangle = \langle f, v \rangle$ for all $v \in W_0^{1,q'}(\Omega)$. Define $h := c$ on $\mathbb{R}^3 \setminus \Omega$, where c is a constant such that

$$\int_D h dx = 0$$

According to Theorem 2.1 in G.P. Galdi *et al.* (1994) there exists a solution $(\tilde{u}, \tilde{p}) \in W^{1,q}(D; \mathbb{R}^3) \times L^q(D)$ of

$$-\Delta \tilde{u} + \nabla \tilde{p} = F, \nabla \times \tilde{u} = h \text{ in } D, \tilde{u} = 0 \text{ on } \partial D.$$

Remark that $(\tilde{u}, \tilde{p}) \in W^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega)$ is a solution of (6), (7). Put $\tilde{g} = \tilde{u}$ on $\partial\Omega$. Then $\tilde{g} \in W^{1-1/q,q}(\partial\Omega; \mathbb{R}^3)$ and

$$\int_{\partial\Omega} n^\Omega \cdot \tilde{g} d\sigma = \int_{\Omega} h dx.$$

Hence

$$\int_{\partial\Omega} n^\Omega \cdot (g - \tilde{g}) d\sigma = \int_\Omega n dx - \int_\Omega h dx = 0.$$

Proposition 3.1 forces that there exists a solution $(w, \theta) \in W^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega)$ of

$$-\Delta w + \nabla \theta = 0, \nabla \cdot w = 0 \text{ in } \Omega, w = g - \tilde{g} \text{ on } \partial\Omega.$$

Put $u := \tilde{u} + w, p := \tilde{p} + \theta$. Then $(u, p) \in W^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega)$ is a solution of (6), (7), (3).

Let now τ and ρ be general. Denote

$$S_{\tau,\rho}(u, p) := -\Delta u(z) + \tau \partial_1 u(z) - [(\rho, 0, 0) \times z] \cdot \nabla u(z) + (\rho, 0, 0) \times u(z) + \nabla p(z).$$

Then $S_{\tau,\rho} : W^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega) \rightarrow W^{-1,q}(\Omega; \mathbb{R}^3)$ is bounded by Theorem 1.4.4.6 and Theorem 1.5.1.2 in P. Grisvard (2011). We show that $S_{\tau,\rho} - S_{0,0}$ is compact. Clearly

$$(S_{\tau,\rho} - S_{0,0})(u, p) = \tau \partial_1 u(z) - [(\rho, 0, 0) \times z] \cdot \nabla u(z) + (\rho, 0, 0) \times u(z).$$

If $v \in C_c^\infty(\Omega; \mathbb{R}^3)$ choose a domain G with smooth boundary such that v is supported in G and $\bar{G} \subset \Omega$. Since $v = 0$ on ∂G , Green's formula gives

$$\begin{aligned} 0 &= \int_{\partial G} v \cdot \{ \tau \partial_1 u - [(\rho, 0, 0) \times z] \cdot \nabla u \} d\sigma(z) \\ &= \int_{\partial G} \{ \tau v \partial_1 u + \tau u \partial_1 v - v \cdot [(\rho, 0, 0) \times z] \cdot \nabla u \} - u \cdot [(\rho, 0, 0) \times z] \cdot \nabla v \} \end{aligned}$$

Thus

$$\langle (S_{\tau,\rho} - S_{0,0})(u, p), v \rangle = \int_\Omega \{ -\tau u \partial_1 v + u \cdot [(\rho, 0, 0) \times z] \cdot \nabla v \} + v \cdot [(\rho, 0, 0) \times u] dz.$$

Density argument forces that this equality holds for arbitrary $v \in W_0^{1,q'}(\Omega; \mathbb{R}^3)$ with $q' = q/(q-1)$. Since $W^{1,q}(\Omega) \rightarrow L^q(\Omega)$ compactly by Lemma 18.4 in L. Tartar (2007), the operator $S_{\tau,\rho} \rightarrow S_{0,0} : W^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega) \rightarrow W^{-1,q}(\Omega; \mathbb{R}^3)$ is compact.

Define $P_{\tau,\rho}(u, p) := (S_{\tau,\rho}(u, p), \nabla \cdot u, \gamma_\Omega u)$. We have proved that

$$P_{0,0} : W^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega) \rightarrow W^{-1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega) \times W^{1-1/q,q}(\partial\Omega; \mathbb{R}^3)$$

is a Fredholm operator with index 0. Since $P_{\tau,\rho} - P_{0,0} = (S_{\tau,\rho} - S_{0,0}, 0, 0)$ and $S_{\tau,\rho} - S_{0,0} : W^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega) \rightarrow W^{-1,q}(\Omega; \mathbb{R}^3)$ is compact, the operator

$$P_{\tau,\rho} : W^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega) \rightarrow W^{-1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega) \times W^{1-1/q,q}(\partial\Omega; \mathbb{R}^3)$$

is a Fredholm operator with index 0, too. We have proved that the dimension of the kernel of $P_{\tau,\rho}$ is equal to 1. So, the co-dimension of the range of $P_{\tau,\rho}$ is also equal to 1. Therefore there exists a solution $(u, p) \in W^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega)$ of (6), (7), (3) if and only if (8) holds.

4. DIRICHLET PROBLEM IN AN EXTERIOR DOMAIN

Now we are going to study the problem (1)–(3) in an exterior domain. We need the following auxiliary results.

Lemma 4.1: Let $\Omega \subset \mathbb{R}^3$ be an unbounded domain with compact smooth boundary, $\tau \neq 0$, $\omega = (\rho, 0, 0)$. Let $1 < q(1) < \infty$. Assume that $f \equiv 0$ and $u \in L^q(1)(\Omega; \mathbb{R}^3)$, $p \in L^1_{loc}(\Omega)$ satisfy (1), (2) in the sense of distributions. Let $2 \leq q(2) < 4$. Fix $r \in (0, \infty)$ such that $\partial\Omega \subset B(0; r)$. Then $u \in D^{1,q(2)}(\Omega \setminus \overline{B(0; r)}; \mathbb{R}^3)$ and there exists $p_\infty \in \mathbb{R}^1$ such that $p - p_\infty \in L^{q(2)}(\Omega \setminus \overline{B(0; r)})$.

(See Lemma 4.1 in D. Kim (2018).)

Lemma 4.2: Let $\Omega \subset \mathbb{R}^3$ be an unbounded domain with compact Lipschitz boundary, $\omega = (\rho, 0, 0)$ and $q, s \in (1, \infty)$. If $(u, p) \in W^{2,q}_{loc}(\Omega, \mathbb{R}^3) \times W^{1,q}_{loc}(\Omega)$, $u \in L^s(\Omega; \mathbb{R}^3)$ is a solution of (1)–(3) with $f \equiv 0$, $g \equiv 0$ satisfying

$$\lim_{r \rightarrow \infty} \int_{\partial B(0;1)} |u(rx)| d\sigma(x) = 0, \quad (9)$$

then $u \equiv 0$ and $\nabla p \equiv 0$.

(See Theorem VIII.7.1 in G. P. Galdi(2011).)

Theorem 4.3: Let $\Omega \subset \mathbb{R}^3$ be an unbounded domain with compact Lipschitz boundary. Suppose that $\tau, \rho \in \mathbb{R}^1 \setminus \{0\}$, $\omega = (\rho, 0, 0)$, $f \in D^{-1,q}(\Omega; \mathbb{R}^3)$ and $g \in W^{1-1/q,q}(\partial\Omega; \mathbb{R}^3)$ with $2 \leq q < 3$. Then there exists a unique solution $(u, p) \in \tilde{W}^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega)$ of (1)–(3).

Proof: Suppose first that $\tau > 0$. Choose an unbounded domain $G \subset \mathbb{R}^3$ with smooth boundary such that $\bar{\Omega} \subset G$. We can consider $D^{1,q'}_0(\Omega)$ to be a closed subspace of $D^{1,q'}_0(G)$. (Here $q' = q/(q-1)$.) According to the Hahn-Banach theorem there exists $F \in D^{-1,q}(G; \mathbb{R}^3)$ such that $\langle F, v \rangle = \langle f, v \rangle$ for all $v \in D^{1,q'}_0(\Omega; \mathbb{R}^3)$. Theorem 2.1 in S. Kracmar *et al.* (QAM 2010) gives that there exists a solution $(w, \theta) \in D^{1,q}_0(G; \mathbb{R}^3) \times L^q(G)$ of

$$-\Delta w(z) + \tau \partial_1 w(z) - (\omega \times z) \cdot \nabla w(z) + \omega \times w(z) + \nabla \theta(z) = F(z), \quad \nabla \cdot w = 0 \text{ in } G.$$

Clearly, $w \in \tilde{W}^{1,q}(\Omega; \mathbb{R}^3)$, $\theta \in L^q(\Omega)$ and (w, θ) is a solution of (1), (2). Moreover, $\gamma_\Omega w \in W^{1-1/q,q}(\partial\Omega; \mathbb{R}^3)$. Put $\tilde{g} := g - \gamma_\Omega w$. Lemma 2.1 gives that $\tilde{g} \in W^{1-1/q,q}(\partial\Omega; \mathbb{R}^3) \subset W^{1/2,2}(\partial\Omega; \mathbb{R}^3)$. According to Theorem VIII.1.2 in G. P. Galdi (2011) there exists $(\tilde{u}, \tilde{p}) \in D^{1,2}(\Omega, \mathbb{R}^3) \times L^2_{loc}(\Omega)$ such that

$$-\Delta \tilde{u}(x) + \tau \partial_1 \tilde{u}(x) - (\omega \times x) \cdot \nabla \tilde{u}(x) + \omega \times \tilde{u}(x) + \nabla \tilde{p}(x) = 0, \quad \nabla \cdot \tilde{u} = 0 \text{ in } \Omega, \quad (10)$$

$$\tilde{u} = \tilde{g} \text{ on } \partial\Omega,$$

$$\int_{\partial B(0;1)} |\tilde{u}(tx)| d\sigma(x) = o(r^{-1/2}) \text{ as } r \rightarrow \infty. \quad (11)$$

Proposition 2.3 gives that $\tilde{u} \in \tilde{W}^{1,2}(\Omega; \mathbb{R}^3)$. Fix $r \in (0, \infty)$ such that $\partial\Omega \subset B(0; r)$ and put $V := \Omega \cap B(0; r)$. Define $\tilde{g} := \tilde{u}$ on $\partial V \setminus \partial\Omega$. Then $\tilde{g} \in W^{1-1/q, q}(\partial V; \mathbb{R}^3)$ because $\tilde{u} \in C^\infty(\Omega; \mathbb{R}^m)$. (See Theorem VIII.1.1 in G.P. Galdi (2011).) According to Theorem 3.2

$$\int_{\partial V} n^V \cdot \tilde{g} \, d\sigma = 0.$$

Applying again Theorem 3.2 we get that there is a solution $(\tilde{v}, \tilde{\pi})$ in the space $W^{1,q}(V; \mathbb{R}^3) \times L^q(V)$ of

$$\begin{aligned} -\Delta \tilde{v}(x) + \tau \partial_1 \tilde{v}(x) - (\omega \times x) \cdot \nabla \tilde{v}(x) + \omega \times \tilde{v}(x) + \nabla \tilde{\pi}(x) &= 0, \quad \nabla \cdot \tilde{v} = 0 \text{ in } V, \\ \tilde{v} &= \tilde{g} \text{ on } \partial V \end{aligned}$$

Hölder's inequality gives that $(\tilde{v}, \tilde{\pi}) \in W^{1,2}(V; \mathbb{R}^m) \times L^2(V)$. Uniqueness result of Theorem 3.2 forces that there exists a constant c such that $\tilde{u} = \tilde{v}$ and $\tilde{p} = \tilde{\pi} + c$. Hence $(\tilde{u}, \tilde{p}) \in W^{1,q}(V; \mathbb{R}^3) \times L^q(V)$.

Since $\tilde{u} \in \tilde{W}^{1,2}(\Omega; \mathbb{R}^3)$, Proposition 2.3 forces that $\tilde{u} \in L^3(\Omega; \mathbb{R}^3)$. Lemma 4.1 gives that there exists a constant p_∞ such that $(\tilde{u}, \tilde{p} - p_\infty) \in D^{1,q}(\Omega \setminus \bar{V}; \mathbb{R}^3) \times L^q(\Omega \setminus \bar{V})$. Since \tilde{u} satisfies (11), we have $\tilde{u} \in \tilde{W}^{1,q}(\Omega \setminus \bar{V}; \mathbb{R}^3)$ by Proposition 2.3.

Put $u := w + \tilde{u}$, $p := \theta + \tilde{p} - p_\infty$. Then $(u, p) \in \tilde{W}^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega)$ is a solution of (1)–(3).

Suppose now that $(u, p) \in \tilde{W}^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega)$ is a solution of (1)–(3) with $f \equiv 0$, $g \equiv 0$. Then $u \in C^\infty(\Omega; \mathbb{R}^3)$ and $p \in C^\infty(\Omega)$ by Theorem VIII.1.1 in G.P. Galdi (2011). Proposition 2.3 forces that u satisfies (9). Moreover, $u \in L^{3/(3-q)}(\Omega; \mathbb{R}^3)$ by Proposition 2.3. Hence $u \equiv 0$ and $\nabla p \equiv 0$ by Lemma 4.2. So, p is constant. Since $p \in L^q(\Omega)$ we infer that $p \equiv 0$.

Let now $\tau < 0$. For $x = [x_1, x_2, x_3]$ denote $\tilde{x} = [-x_1, x_2, x_3]$. Put $\tilde{\Omega} := \{\tilde{x}; x \in \Omega\}$. For a function φ defined on Ω define a function $\tilde{\varphi}$ on $\tilde{\Omega}$ by

$$\tilde{\varphi}(\tilde{x}) := \varphi(x).$$

Remark that $\tilde{p} \in L^q(\tilde{\Omega})$ if and only if $p \in L^q(\Omega)$, and $\tilde{u} \in \tilde{W}^{1,q}(\tilde{\Omega}; \mathbb{R}^3)$ if and only if $u \in \tilde{W}^{1,q}(\Omega; \mathbb{R}^3)$. Define $\tilde{g}(x) := g(\tilde{x})$ for $x \in \partial\tilde{\Omega}$, and $\langle \tilde{f}, v \rangle := \langle f, \tilde{v} \rangle$ for $v \in D_0^{1,q'}(\tilde{\Omega}; \mathbb{R}^3)$ with $q' = q/(q-1)$. Clearly, (u, p) is a solution of (1)–(3) if and only if (\tilde{u}, θ) with $\theta := -\tilde{p}$ is a solution of

$$\begin{aligned} -\Delta \tilde{u}(z) - r \partial_1 \tilde{u}(z) - (\omega \times z) \cdot \nabla \tilde{u}(z) + \omega \times \tilde{u}(z) + \nabla \theta(z) &= \tilde{f}(z), \\ \nabla \cdot \tilde{u} &= 0 \text{ in } \tilde{\Omega}, \\ \tilde{u} &= \tilde{g} \text{ on } \partial\tilde{\Omega}. \end{aligned}$$

Therefore there exists a unique solution $(u, p) \in \tilde{W}^{1,q}(\Omega; \mathbb{R}^3) \times L^q(\Omega)$ of (1)–(3).

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